We will frequently use the following topics in the course. We will only cover the minimal amount of content. These are necessary for your understanding of the course.

A. Trigonometry

First of all, the angles in the course are in radians, unless noted otherwise. Recall that radians are scaled so that 180° is π radians.

<u>Example</u>

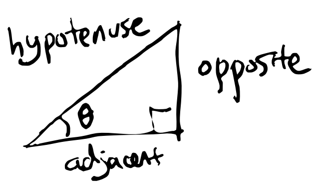
$45^\circ = \frac{\pi}{4}$ radians

$60^\circ = \frac{\pi}{3}$ radians

$90^\circ = \frac{\pi}{2}$ radians

$270^\circ = \frac{3\pi}{2}$ radians.

Trigonometric functions are defined as follows.



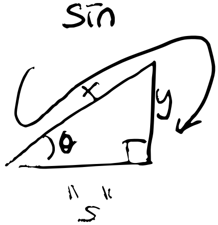
$\sin\theta = \frac{opp}{hyp}$ (sine)

$\cos\theta = \frac{adj}{hyp}$ (cosine)

$\tan\theta = \frac{opp}{adj}$ (tangent)

Mnemonics: write the first letter, and the value is $\frac{\text{second side}}{\text{first side}}$.

sin

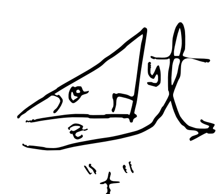


$\frac{y}{x}$

cos

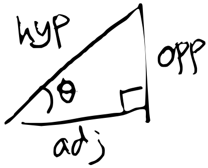
$\frac{z}{x}$

tan



$\frac{y}{z}$

Some more trig functions:



$$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj} \quad (\text{secant})$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp} \quad (\text{cosecant})$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp} \quad (\text{cotangent})$$

In polar coordinates $(r,\theta) = (1,\theta)$

$\updownarrow$

In rectangular coordinates $(x,y) = (\cos\theta, \sin\theta)$.

(Then $\tan\theta = \frac{y}{x}$

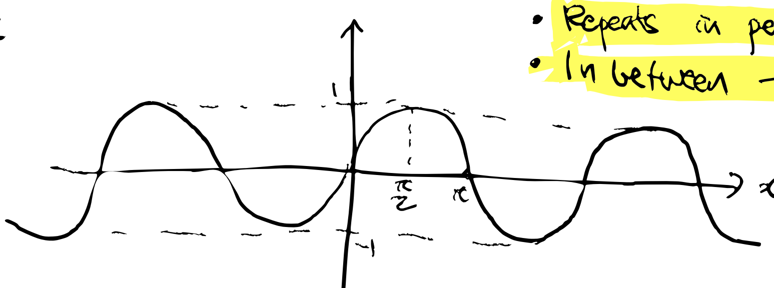
$\sec\theta = \frac{1}{x}$

$\csc\theta = \frac{1}{y}$

$\cot\theta = \frac{x}{y}$.)

The graphs of trig functions:

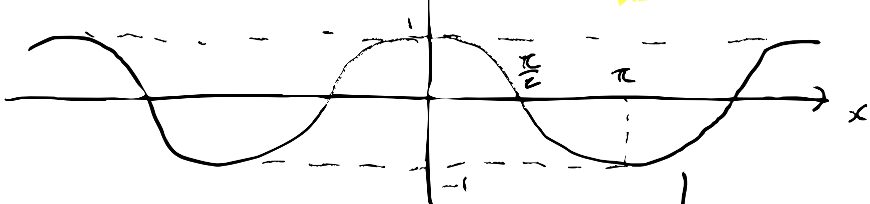
$f(x) = \sin x$



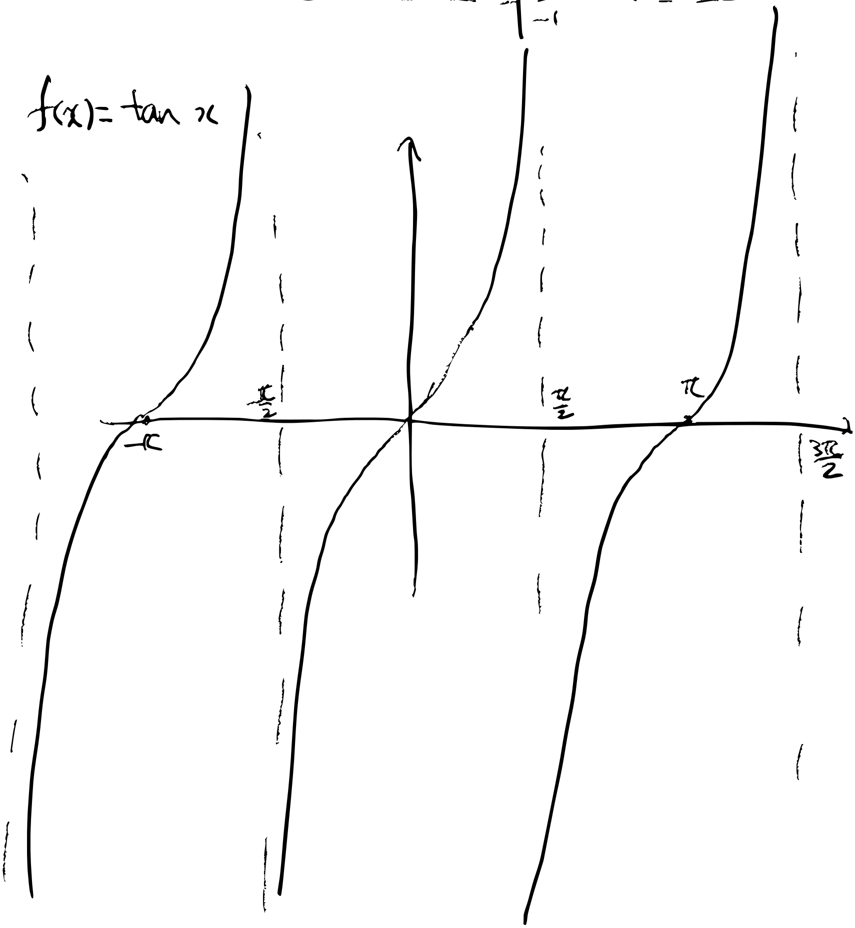
- Repeats in period of 2π
- In between -1 and 1.

$f(x) = \cos x$

- Repeats in period of 2π
- In between -1 and 1

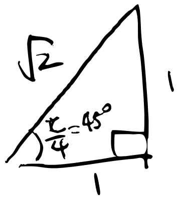


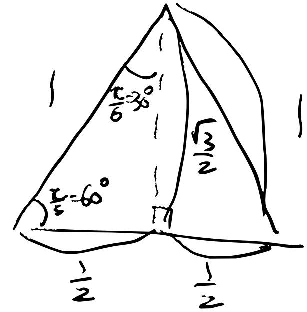
$f(x) = \tan x$



- Repeats in period of π
- Blows up at $\frac{\pi}{2}, \frac{3\pi}{2}, \ldots, -\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots$

Some special triangles:





(half of the equilateral triangle)

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan\frac{\pi}{4} = 1$

$\cos\frac{\pi}{3} = \frac{1}{2}$ $\quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \sin\frac{\pi}{6} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \sqrt{3}$ $\quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\cos -\pi = -1$ $\quad \cos -\frac{\pi}{2} = 0$

$\cos 0 = 1$ $\quad \cos\frac{\pi}{2} = 0$

$\cos \pi = -1$ $\quad \cos\frac{3\pi}{2} = 0$

$\sin -\pi = 0$ $\quad \sin -\frac{\pi}{2} = -1$

$\sin 0 = 0$ $\quad \sin\frac{\pi}{2} = 1$

$\sin \pi = 0$ $\quad \sin\frac{3\pi}{2} = -1$

$\tan 0 = \tan \pi = \cdots = 0$

$\tan\frac{\pi}{2}, \tan -\frac{\pi}{2}, \ldots$ not defined

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}$

$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$

$\tan\frac{3\pi}{4} = -1$ $\quad \tan\frac{5\pi}{4} = 1$ $\quad \tan\frac{7\pi}{4} = -1$

Some identities:

- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- $\cos^2\theta + \sin^2\theta = 1$ ($\cos^2\theta$ is $(\cos\theta)^2$, for example)

- $\cos(-\theta) = \cos\theta$
- $\cos(\theta + \pi) = -\cos\theta$
- $\cos(\theta + 2\pi) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\sin(\theta + \pi) = -\sin\theta$
- $\sin(\theta + 2\pi) = \sin\theta$
- $\tan(-\theta) = -\tan\theta$
- $\tan(\theta + \pi) = \tan\theta$
- $\sec^2\theta = \tan^2\theta + 1$

$\cos\frac{2\pi}{3} = -\frac{1}{2}$ $\quad$ $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad$ $\sin\frac{5\pi}{6} = \frac{1}{2}$

$\tan\frac{2\pi}{3} = -\sqrt{3}$ $\quad$ $\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$ $\quad$ $\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\sin\frac{7\pi}{6} = -\frac{1}{2}$ $\quad$ $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad$ $\tan\frac{4\pi}{3} = \sqrt{3}$

$\cos\frac{5\pi}{3} = \frac{1}{2}$ $\quad$ $\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$ $\quad$ $\sin\frac{11\pi}{6} = -\frac{1}{2}$

$\tan\frac{5\pi}{3} = -\sqrt{3}$ $\quad$ $\tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$.

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

B. Completing the squares

A quadratic (in x) is an expression of the form

ax^2+bx+c. Here x is the variable, whereas

a, b, c are constants.

<u>Example</u> $2x^2+6x-1$

$3x^2$

$-x^2+1$

$\vdots$

A quadratic can be obtained by multiplying two

linear forms,

$$(kx+l)(mx+n) = kmx^2+(lm+kn)x+ln.$$

Finding such expression is called factorization.

So if you were to solve a quadratic equation,

$0 = ax^2+bx+c$, you can try to factorize it.

<u>Example</u> The solution of $x^2-4x+3=0$ is either $x=1$

or $x=3$, because $(x-1)(x-3) = x^2-4x+3$.

In general $ax^2+bx+c=0$ has solutions given by

the quadratic formula,

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}.$$

(so either + or −).

This is obtained by completing the square.

Namely, we know

$$(kx+l)^2 = k^2x^2 + 2klx + l^2.$$

So

$$a\left(x+\frac{b}{2a}\right)^2 = a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)$$

$$= ax^2+bx+\frac{b^2}{4a},$$ which matches

with ax^2+bx+c except the last

term.

So

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right).$$

So $0 = ax^2 + bx + c$

$$\Rightarrow \quad 0 = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

$$\Rightarrow \quad \frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \quad \frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \quad \frac{\pm\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Warning

- $\sqrt{\text{something}}$ is a bit subtle.
  - There are usually two numbers whose squares are the given number, but $\sqrt{\ }$ only takes the positive one.

    Example $2^2 = 4 = (-2)^2$, but $\sqrt{4}$ is 2, not -2.
- You can't take $\sqrt{\ }$ of a negative number, unless you learn the complex numbers (see optional materials). There, $\sqrt{\ }$ becomes really ambiguous.

C. Solving a system of linear equations.

A typical problem is:

find the values of x and y satisfying the following two equations.

$$2x - y = 3$$

$$x + 3y = 5.$$

The only strategy: try to eliminate variables one by one. If it works, that's great!

The case of $\begin{matrix} 2x - y = 3 & Ⓐ \\ x + 3y = 5 & Ⓑ \end{matrix}$:

$2 \times$ Ⓑ is $2x + 6y = 10$. so the same $2x$ appears as in Ⓐ.

2Ⓑ $-$ Ⓐ is $(2x + 6y) - (2x - y) = 10 - 3$, or

$$7y = 7, \text{ so } y = 1.$$

Plug this back into the equations. For example, Ⓐ becomes

$2x - 1 = 3$, $2x = 4$, or $x = 2$. Done! $x = 2, y = 1$.

You'll learn a slightly more streamlined way in a linear algebra class, but the gist is what I've just explained. There is no secret recipe.

- You try to eliminate variables by fiddling around.
- Hopefully you get the value of a variable.
- Plug this value back into the original equations.

  Now you have one less variable, so repeat the above process until you know the values of all variables.

Example Solve x, y, z for

$$6x - y + 5z = 9 \quad \text{(A)}$$
$$-x - y + 3z = -5 \quad \text{(B)}$$
$$2x + 3y - z = 13. \quad \text{(C)}$$

Solution First objective: Eliminate x as much as possible.

$2 \times$ (B) is $-2x - 2y + 6z = -10$, so

2(B) + (C) is $(-2x - 2y + 6z) + (2x + 3y - z) = -10 + 13$,

or

$$y + 5z = 3. \quad \text{(D)}$$

$6 \times$ B is $-6x - 6y + 18z = -30$, so

6B $+$ A is $(-6x - 6y + 18z) + (6x - y + 5z) = -30 + 9$,

or $-7y + 23z = -21$. E

We may then try to solve

$$y + 5z = 3 \qquad \text{D}$$
$$-7y + 23z = -21 \qquad \text{E}$$

Subobjective Eliminate y as much as possible.

$7 \times$ D is $7y + 35z = 21$, so

7D $+$ E is $(7y + 35z) + (-7y + 23z) = 21 - 21$, or

$58z = 0$, or

$z = 0$.

So we've figured out one variable! Two more to go.

Plug $z=0$ back into the original equations:

$$6x - y = 9 \qquad \text{A}$$
$$-x - y = -5 \qquad \text{B}$$
$$2x + 3y = 13 \qquad \text{C}$$

Again,

Objective Eliminate x as much as possible.

$2\times$ B is $-2x-2y=-10$, so

2 B $+$ C is $(-2x-2y)+(2x+3y)=-10+13$, or

$y=3$. Yes!

Plug this back into the original equations:

$6x=12$ A

$-x=-2$ B

$2x=4$ C

$\Rightarrow$ $x=2$, and we're done.

$x=2$
$y=3$
$z=0$.

Note 1. A general intuition is that if you have n many variables and n many equations, the system is usually solvable, and has a unique (one and only one) solution.

Of course, this is not always true.

Example $x+y=4$ (A)

$2x+2y=1$ (B)

has no solution, because 2(A) is $2x+2y=8$, while (B) is $2x+2y=1$,

which can't be true simultaneously.

Example $x+y=4$ (A)

$2x+2y=8$ (B)

has infinitely many solutions, because

2(A) is $2x+2y=8$,

(B) is $2x+2y=8$,

so (B) doesn't give you any more new information.

So $x=0$, $y=4$
$x=1$, $y=3$
$\vdots$
are all possible.

2. All linear systems of equations fall into one of the following three categories.

- No solutions (contradicting equations)
- Infinitely many solutions (not enough equations)
- Unique solution.

If you have too few equations, the chances are that the system has infinitely many solutions.
(too few equations ⇝ not enough information)
If this happens, we call it underdetermined.

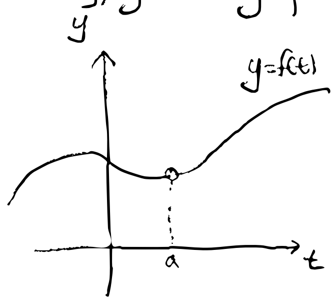
If you have too many equations, the chances are that the system has no solutions.
(too many equations ⇝ contradicting information)
If this happens, we call it overdetermined.

D. Limits

For a function $f(t)$,

$\lim_{t\to a} f(t) = L$ means that ==f(t) gets close to L as t gets close to L.==

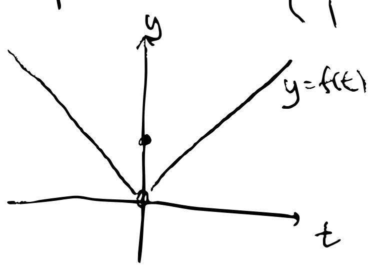
Namely, given a graph that is missing a point over $t=a$,



what is the point that can ~~se~~ seamlessly connect it?

If $\lim_{t\to a} f(t) = f(a)$, we say f is ==continuous at t=a.==

<u>Example</u> Let $f(t) = \begin{cases} t & t \neq 0 \\ 1 & t = 0. \end{cases}$ Then the graph looks like

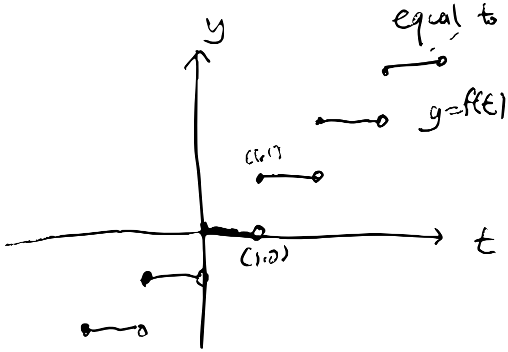


So even though the actual value of $f(0)=1$, the value that can seamlessly connect the graph at $t=0$ is

$\lim_{t\to 0} f(t) = 0.$

So f is not continuous at $t=0$

Example $f(t) = \lfloor t \rfloor$ (the largest integer that is smaller than or equal to t) has graph



Then at $t=1$, there is no way that the two sides can be connected seamlessly, so this is not continuous at $t=1$.

Functions that are continuous everywhere:

- Polynomials
- e^t
- $\cos t$, $\sin t$
- Any function that is obtained by using $+, -, \times$, composition of the above

(Examples: $\cos^2 t + \cos t - 1 + \sin^3 t + e^t \cos t - t^2 - e^{t^2} \cos(t^2+1)$

$\sin(\cos(e^{t^2+3}))$ . . .)

The advantage of lim is that you can consider the "supposed" value at $t=a$ even if $f(t)$ is not defined there. This applies to the points right at the boundary of domain.

<u>Example</u> $\lim_{t\to\infty} \frac{1}{t} = 0$, since $\frac{1}{t}$ approaches 0 (gets smaller and smaller) as t gets bigger.

Similarly, $\lim_{t\to-\infty} \frac{1}{t} = 0$, since $\frac{1}{t}$ approaches 0 (it's − of a smaller and smaller number) as t gets − of a bigger number.

<u>Example</u> $\lim_{t\to\infty} t^2 = \infty$ because t^2 gets larger as t gets larger.

Similarly $\lim_{t\to-\infty} t^2 = \infty$.

$\lim_{t\to-\infty} t = -\infty$ because t is − of a bigger number as t becomes − of a bigger number (tautology)

<u>Warning</u> ∞ or $-\infty$ is not really a number, so you need to perform $+, \times, -, \div$ carefully with these.

Basically

$$\boxed{\begin{array}{ll} \infty + (\text{any number}) = \infty & \infty - (\text{any number}) = \infty \\ \infty \times (\text{any number} > 0) = \infty & \infty \times (\text{any number} < 0) = -\infty \\ \infty \div (\text{any number} > 0) = \infty & \infty \div (\text{any number} < 0) = -\infty \\ \infty \times \infty = \infty & \infty \times (-\infty) = -\infty \\ \infty + \infty = \infty & \infty - (-\infty) = \infty \quad \ldots \end{array}}$$

But some operations are inconclusive (can be anything), so you shouldn't use them.

Inconclusive operations:

$\infty - \infty$	✗
$\infty \times 0$	✗
$\infty \div \infty$	✗
$\infty \div 0$	✗

Similar analogues hold for $-\infty$.

This can be useful because if lim exists, lim can be computed before/after $+, \times, -, \div$ and composition

Namely,

1. If $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ exists,

$$\lim_{t\to a}\left(f(t)+g(t)\right) = \lim_{t\to a} f(t) + \lim_{t\to a} g(t)$$

$$\lim_{t\to a}\left(f(t)\,g(t)\right) = \left(\lim_{t\to a} f(t)\right)\cdot\left(\lim_{t\to a} g(t)\right)$$

$$\lim_{t\to a}\left(f(t)-g(t)\right) = \lim_{t\to a} f(t) - \lim_{t\to a} g(t)$$

(If $\lim_{t\to a} g(t) \neq 0$)

$$\lim_{t\to a}\frac{f(t)}{g(t)} = \frac{\lim_{t\to a} f(t)}{\lim_{t\to a} g(t)}$$

(If you follow the warning above, you can also apply arithmetic including ∞ and $-\infty$.)

2. If $\lim_{t \to a} g(t)$ exists and is L, and

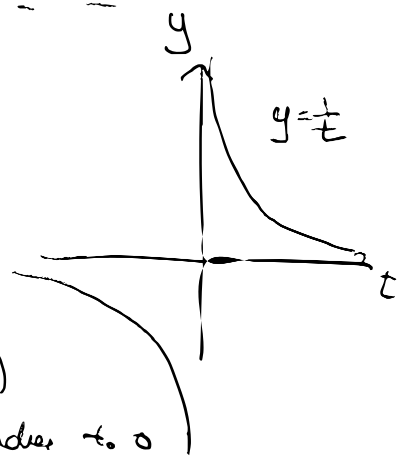
$\lim_{s \to L} f(s)$ exists, then

$$\lim_{t \to a} f(g(t)) = \lim_{s \to L} f(s) = \lim_{s \to \lim_{t \to a} g(t)} f(s).$$

If f is continuous at $s = L$, this means

$$\lim_{t \to a} f(g(t)) = f(\lim_{t \to a} g(t)).$$

Sometimes lim does not exist.

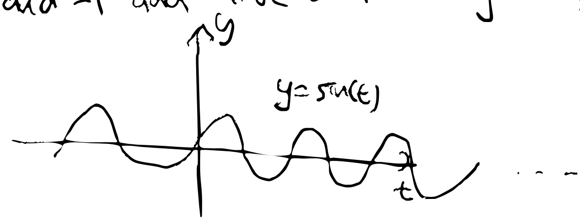


<u>Example</u> $\lim_{t \to 0} \frac{1}{t}$ does not exist because

it approaches to ∞ if t approaches to 0 from the right (using positive numbers) and it approaches to $-\infty$ if t approaches to 0 from the left (using negative numbers).

<u>Example</u> $\lim_{t \to \infty} \sin t$ does not exist because

it oscillates between 1 and -1 and there is no tendency of approaching to a value.



E. Derivatives

meaning continuous at all $t \in (a-\varepsilon, a+\varepsilon)$ for some $\varepsilon > 0$

For a function $f(t)$ that is continuous at around $t=a$,

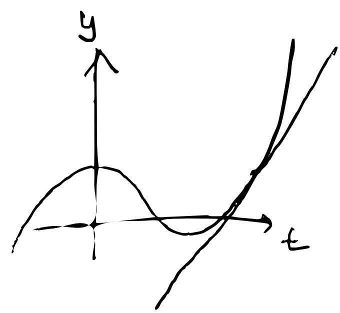
we say f is differentiable at $t=a$ if

$\lim_{t\to a} \frac{f(t)-f(a)}{t-a}$ exists. If it exists, we call this

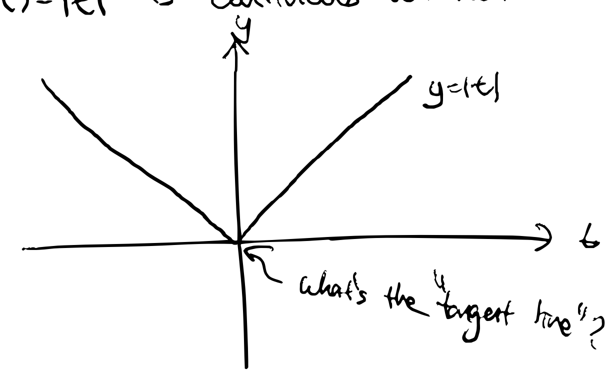
quantity $f'(a) := \lim_{t\to a} \frac{f(t)-f(a)}{t-a}$.

- Tangent line at $t=a$ is

  $y = f'(a)(t-a) + f(a)$



- $f(t) = |t|$ is continuous but not differentiable at $t=0$.



what's the "tangent line"? ambiguous, so not differentiable.

- not defined at $t=a$ $\Rightarrow$ can't be continuous at $t=a$.

  not continuous at $t=a$ $\Rightarrow$ can't be differentiable at $t=a$.

Basically all functions you'll encounter in a natural setting would be differentiable over the whole domain except a few points of singularities. So you think the derivative $f'(t)$ as another function. Sometimes you write $\frac{df}{dt}$ for this.

$f(t) = t^n \rightarrow f'(t) = nt^{n-1}$

$f(t) = \cos t \rightarrow f'(t) = -\sin t$

$f(t) = \sin t \rightarrow f'(t) = \cos t$

$f(t) = e^t \rightarrow f'(t) = e^t$

$f(t) = \ln t \rightarrow f'(t) = \frac{1}{t}$ $(t>0)$

$f(t) = g(t)h(t) \rightarrow f'(t) = g'(t)h(t) + g(t)h'(t)$ (Product Rule)

$f(t) = g(t) + h(t) \rightarrow f'(t) = g'(t) + h'(t)$

$f(t) = g(t) - h(t) \rightarrow f'(t) = g'(t) - h'(t)$

$f(t) = \frac{g(t)}{h(t)} \rightarrow f'(t) = \frac{g'(t)h(t) - g(t)h'(t)}{h(t)^2}$ (Division Rule)

$f(t) = g(h(t)) \rightarrow f'(t) = g'(h(t))\, h'(t)$ (Chain Rule)

(Often written as "$\frac{df}{dt} = \frac{df}{dh}\frac{dh}{dt}$.")

If $y = f(t)$ is the inverse function of $t = g(y)$, (namely $f(g(y)) = y$, $g(f(t)) = t$)

then $f'(t) = \frac{dy}{dt} = \frac{1}{\frac{dt}{dy}} = \frac{1}{g'(y)} = \frac{1}{g'(f(t))}$.

The rest is the combination of the above.

F. L'Hôpital's Rule

Whenever you're finding the limit

$\lim_{t \to a} \frac{f(t)}{g(t)}$, the first thing you would do is to put $t=a$ into the expression and get $\frac{f(a)}{g(a)}$.

Each $f(a)$. can fall into three categories:

- $f(a) = 0$
- $f(a) =$ nonzero number
- $f(a) = \infty$ (this means either $+\infty$ or $-\infty$).

The same applies to $g(a)$.

Then, most combinations give the answer, except $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

$g(a)$ \ $f(a)$	0	nonzero #	∞
0	$\frac{f(a)}{g(a)} = \frac{0}{0}$ (?)	$\frac{f(a)}{g(a)} = \frac{\text{sth}}{0} = \infty$	$\frac{f(a)}{g(a)} = \frac{\infty}{0} = \infty$
nonzero #	$\frac{f(a)}{g(a)} = \frac{0}{\text{sth}} = 0$	$\frac{f(a)}{g(a)} = \frac{\text{sth}}{\text{sth}}$	$\frac{f(a)}{g(a)} = \frac{\infty}{\text{sth}} = \infty$
∞	$\frac{f(a)}{g(a)} = \frac{0}{\infty} = 0$	$\frac{f(a)}{g(a)} = \frac{\text{sth}}{\infty} = 0$	$\frac{f(a)}{g(a)} = \frac{\infty}{\infty}$ (?)

If $\frac{f(a)}{g(a)}$ is of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then you're allowed to use L'Hôpital's Rule,

$$\lim_{t\to a} \frac{f(t)}{g(t)} = \lim_{t\to a} \frac{f'(t)}{g'(t)}.$$

Then you have a new limit, and start the same procedure again, in hopes that at some stage you get a limit that you could compute via other means.

Ex $\lim_{t\to 0} \frac{e^{t^2}-1}{t^2} = ?$ This is $\frac{0}{0}$, so by L'Hôpital

$$\lim_{t\to 0} \frac{e^{t^2}-1}{t^2} = \lim_{t\to 0} \frac{2te^{t^2}}{2t} = \lim_{t\to 0} e^{t^2} = e^0 = 1.$$

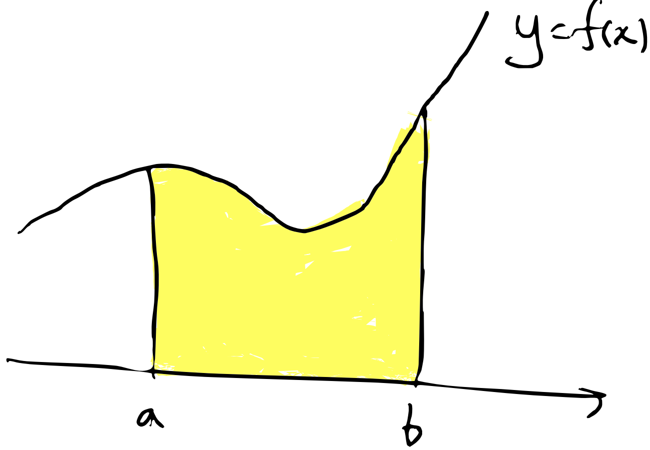
Ex $\lim_{t\to 0} \frac{e^{t}-1-t}{t^2} = ?$ this is $\frac{0}{0}$, so by L'Hôpital

$\lim_{t\to 0} \frac{e^{t}-1-t}{t^2} = \lim_{t\to 0} \frac{e^{t}-1}{2t}$. This is again $\frac{0}{0}$, so again by L'Hôpital,

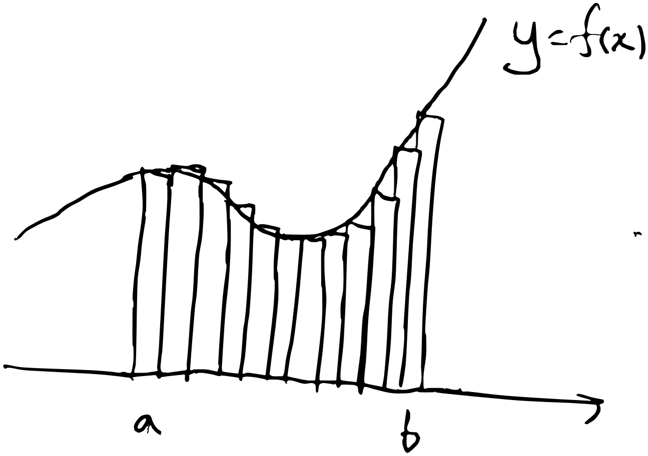
$$\lim_{t\to 0} \frac{e^{t}-1}{2t} = \lim_{t\to 0} \frac{e^{t}}{2} = \frac{e^0}{2} = \frac{1}{2}.$$

G. Integrals.

The textbook definition of $\int_a^b f(x)\,dx$ is the area of the region formed by the graph $y=f(x)$



and this is obtained by Riemann sum.



$\sim$ ... smaller rectangles

so $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right)$$

In reality, you use Fundamental Theorem of Calculus,

$$f(x) = g'(x) \longrightarrow \int f(x)\,dx = g(x) + C$$

or $$\int_a^b f(x)\,dx = g(b) - g(a)$$

Here, $\int$ without a & b is called the indefinite integral. For indefinite integrals, you always put an unspecified constant C

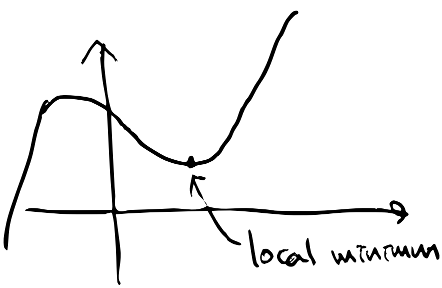
Finding integrals can be done by looking at the table of derivatives and reversing the arrows.

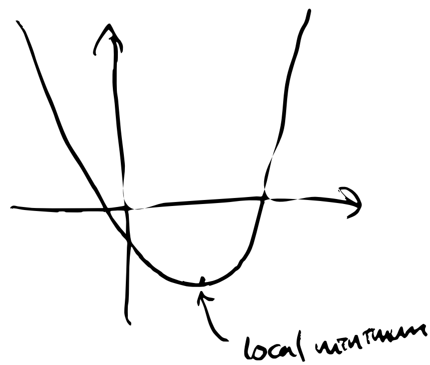
There are more sophisticated tools like substitution and integration by parts which we don't need.

H. Finding min/max

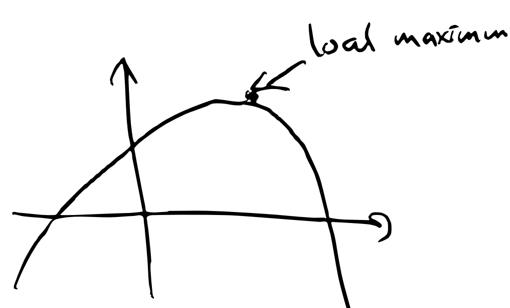
For a function $f(t)$, a critical point is t such that $f'(t)=0$.

A local minimum is t such that $f(t)$ is smaller than or equal to any other values of f around $f(t)$

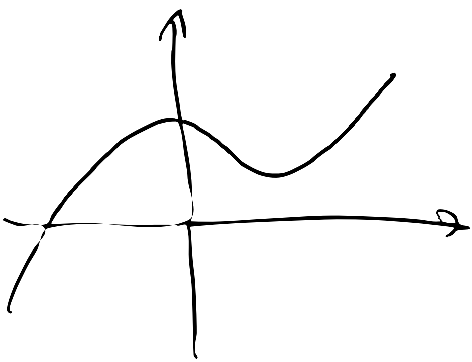


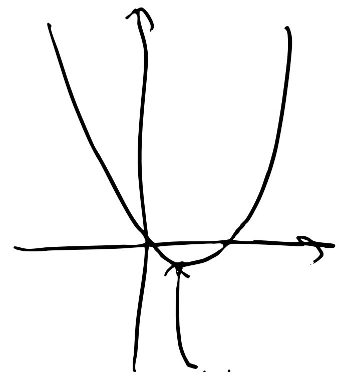


A local maximum is similar, but $f(t)$ has to be bigger than or equal to its neighboring values.



A global minimum is t such that $f(t)$ is the smallest possible value of f.





Global maxima are similar.

The process of finding local/global max/min differs quite a lot depending on how the domain looks like.

Ⓐ If the domain is $[a,b]$ for a, b numbers:

local/global max/min are among
- critical points
- boundary points (namely, a and b).

Ex Find the local/global max/min of $f(x)=x^2$ at $[-1,2]$.

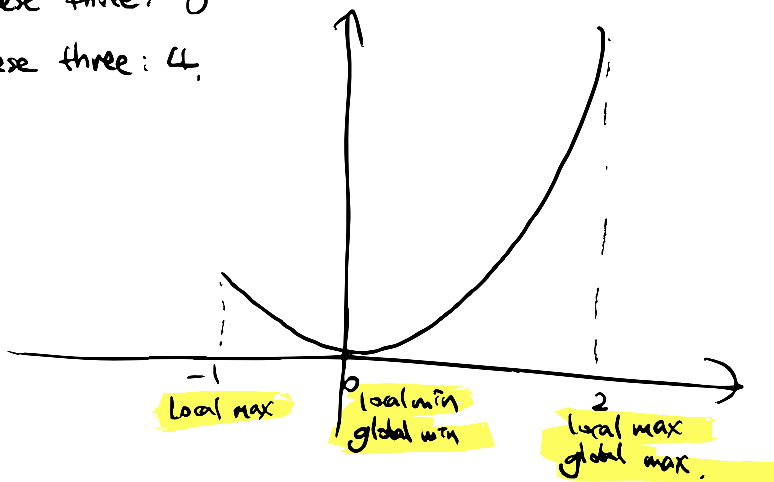
Critical points: $f'(x)=0$, so $x=0$. $\Rightarrow f(0)=0$.

Boundary: $f(-1)=1$

$f(2)=4$.

Min of these three: 0

Max of these three: 4.



(B) If the domain is $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$.

Local min/max are among
- Critical points
- Boundary points

$[a,\infty)$: a

$(-\infty,b]$: b

$(-\infty,\infty)$: none

Global min/max are among them, but you also have to compare their values with $\lim_{x\to\infty} f(x)$ (if ∞ appears in the domain) and $\lim_{x\to-\infty} f(x)$ (if $-\infty$ appears in the domain)

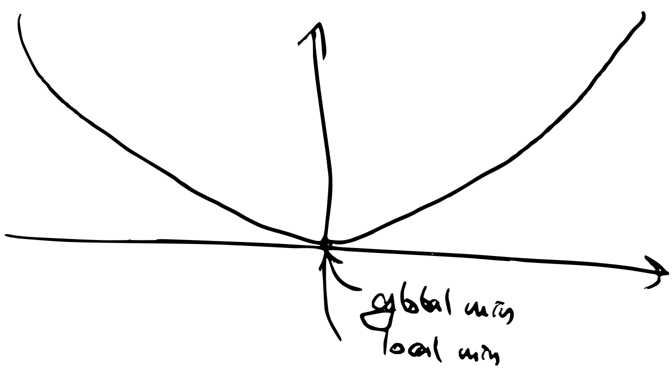
<u>Ex</u> $f(x)=x^2$ at $(-\infty,\infty)$.

Critical points: $f'(x)=0$, so $x=0 \Rightarrow f(x)=0$

No boundary points

$\lim_{x\to+\infty} f(x)=+\infty$ $\lim_{x\to-\infty} f(x)=+\infty$

Among these, min: 0, max: $+\infty$, so there is only global minimum and not maximum.



<u>Ex</u> $f(x)=x^3-3x$ at $(-\infty,3]$

Critical points: $f'(x)=3x^2-3=0$, so $x=+1$ or -1

$f(1)=-2$

$f(-1)=2$

Boundary point: $x=2 \rightarrow f(2)=2$

$\lim_{x\to-\infty} f(x) = -\infty$.

So among $\{-2, 2, 2, -\infty\}$, max is 2, min is $-\infty$, only achieved via limit

actual values of f

lim of f

So there is no global min.